Deriving informative prior distributions by the combination of the opinions of several experts in a Bayesian hierarchical approach. An application in radiation epidemiology.

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The use of prior distributions that encode external information is limited in Bayesian inference.
Background

- The use of prior distributions that encode external information is limited in Bayesian inference

Possible reasons:
- Lack of domain knowledge
- Ensure a certain “objectivity” of analyses
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In complex hierarchical models it may be indispensable to derive informative prior distributions for parameters that are only poorly informed by the data
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⇒ Elicit expert opinion to derive prior distributions
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- Ensure a certain “objectivity” of analyses
- In complex hierarchical models it may be indispensable to derive informative prior distributions for parameters that are only poorly informed by the data

⇒ Elicit expert opinion to derive prior distributions

⇒ Challenges:
- Develop an elicitation procedure that allows to derive a prior distribution reflecting an expert’s knowledge
- Combine the information of several experts to derive a unique prior distribution
Ask questions that the expert can understand and answer
Strategies to avoid cognitive biases in the elicitation process

- Ask questions that the expert can understand and answer
  - Prefer questions concerning observable quantities
  - Quantiles are easier to assess than variances
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  ⇒ It may be easier to derive information based on the expert’s choices
Strategies to avoid cognitive biases in the elicitation process

- Ask questions that the expert can understand and answer
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- Difficult to give direct information concerning parameter uncertainty
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- Experts should be given a training in the elicitation procedure
- Give visual feedback
- Assess the confidence an expert has in his answers
Motivating example: Dose estimation in uranium miners

Measurement error

Radon

Uncertainties

breathing characteristics

mine atmosphere

type of ventilation

absorbed lung dose
Accounting for exposure uncertainty

\[
\lambda, \beta, V_i(t), D_i^{\text{cum}}(t), D_i(t), Y_i, \delta_i, \theta_i(t), \mu_X, \sigma_X, \sigma_{\epsilon^B}, \sigma_{\epsilon^C}, X_i(t), Z_i^B(t), Z_i^C(t)
\]

dose

additional covariates

cumulated dose
time until death by lung cancer

true annual exposure

observed annual exposure (Berkson period)

observed annual exposure (classical period)
Integrating dose uncertainties

- $\lambda$ (
- $\beta$
- $V_i(t)$
- $D_i^{cum}(t)$
- $\gamma_i, \delta_i$
- time until death by lung cancer
- cumulated dose

- $\theta_i(t)$
- $\mu_X$
- $\sigma_X$
- $\sigma_{\epsilon}^B$
- $\sigma_{\epsilon}^C$
- $Z_i^B(t)$
- observed annual exposure (Berkson period)

- $X_i(t)$
- true annual exposure

- $Z_i^C(t)$
- observed annual exposure (classical period)
Sensitivity analyses show that the most important parameters intervening in dose calculation are:

- Biological inter-subject variability:
  - Breathing rate
  - Fraction breathed through nose
  - Thickness of the bronchial epithelium

- Activity size distribution

- Unattached fraction
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  - Fraction breathed through nose
  - Thickness of the bronchial epithelium

- Activity size distribution

- Unattached fraction
Breathing rate

- Average breathing rate of miner $i$ at time $t$ can be determined by multiplying the proportion of time a miner spent in a certain activity by the corresponding value for breathing rate:

$$\bar{br}_i(t) = p_1(i, t) \cdot br_1 + p_2(i, t) \cdot br_2 + p_3(i, t) \cdot br_3$$
Breathing rate

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- Reference values commonly used in radiation protection are based on data on six miners in a metal mine in Tadjikistan and on 620 miners in a gold mine in South Africa [Birchall and James, 1994, Ruzer et al., 1995]

- ICRP publication 66:
  1.0 - 1.7 m$^3$h$^{-1}$

- Alpha risk project:
  0.9 - 1.7 m$^3$h$^{-1}$
  1.1 - 2.1 m$^3$h$^{-1}$
Breathing rate

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⇒ Ask experts about the time a miner spent sitting, in light activity and in heavy activity given a workday of eight hours

Six conditions: Hewers, underground and open pit miners and before and after mechanisation of the mines.
The Elicitation procedure
Vous gagnez 20 euros si

On tourne la roue
et elle s'arrête sur bleue

La température maximale sera inférieure à 8.5 degrés
Choix des conditions de travail

- Pour un foreur avant la mécanisation
- Pour un mineur de fond avant la mécanisation
- Pour un mineur de jour avant la mécanisation

- Pour un foreur après la mécanisation
- Pour un mineur de fond après la mécanisation
- Pour un mineur de jour après la mécanisation
For each condition, the expert \( e \ (e \in \{1, 2, 3\}) \) is presented with a series of choices concerning the time a worker in this condition spent sitting, in light exercise and in heavy exercise.
Vous gagnez 20 euros si

On tourne la roue et elle s'arrête sur bleue

Le temps assis pour un mineur de fond après la mécanisation est inférieur à 1 heures et 30 minutes

Choisir

Indécis
Vous gagnez 20 euros si

On tourne la roue et elle s'arrête sur bleue

Choisir

Indécis
Revenir en arrière

Le temps assis pour un mineur de fond après la mécanisation est inférieur à 0 heures et 30 minutes

Choisir
Vous gagnez 20 euros si

On tourne la roue et elle s'arrête sur bleue

Le temps assis pour un mineur de fond après la mécanisation est inférieur à 2 heures et 15 minutes
The elicitation procedure

- For each condition, the expert \( e \) (\( e \in \{1, 2, 3\} \)) is presented with a series of choices concerning the time a worker in this condition spent sitting, in light exercise and in heavy exercise.

- Based on these binary choices, we derive the first quartile \( q_{ej}^{0.25} \), the median \( q_{ej}^{0.5} \) and the third quartile \( q_{ej}^{0.75} \) of expert \( e \) and variable \( j \), \( j \in \{1, 2, 3\} \).

- Using an interval-halving algorithm and a least squares method, we fit two alternative beta distributions based on \( q_{ej}^{0.25} \) and \( q_{ej}^{0.5} \), and \( q_{ej}^{0.75} \) and \( q_{ej}^{0.5} \), respectively.
Quelle confiance avez-vous dans votre choix pour le temps avec une forte activité physique ?
The elicitation procedure

- For each condition, the expert $e$ ($e \in \{1, 2, 3\}$) is presented with a series of choices concerning the time a worker in this condition spent sitting, in light exercise and in heavy exercise.
- Based on these binary choices, we derive the first quartile $q_{ej}^{0.25}$, the median $q_{ej}^{0.5}$ and the third quartile $q_{ej}^{0.75}$ of expert $e$ and variable $j$, $j \in \{1, 2, 3\}$.
- Using an interval-halving algorithm and a least squares method, we fit two alternative beta distributions based on $q_{ej}^{0.25}$ and $q_{ej}^{0.5}$, and $q_{ej}^{0.75}$ and $q_{ej}^{0.5}$, respectively.
- The expert is presented with two alternative histograms representing the two fitted Beta distributions.
- He expresses his confidence in these distributions via $c_{ej}^{0.25}$ and $c_{ej}^{0.75}$ taking values between 1 and 9.
Fitting a distribution that reflects an expert’s knowledge

For each condition, we dispose of 9 quantiles and 6 evaluations for an expert $e$ at the end of the elicitation process:

- **Sitting:** $q_{e_1}^{0.25}, q_{e_1}^{0.5}, q_{e_1}^{0.75}$ with evaluations $c_{e_1}^1, c_{e_1}^2$
- **Light exercise:** $q_{e_2}^{0.25}, q_{e_2}^{0.5}, q_{e_2}^{0.75}$ with evaluations $c_{e_2}^1, c_{e_2}^2$
- **Heavy exercise:** $q_{e_3}^{0.25}, q_{e_3}^{0.5}, q_{e_3}^{0.75}$ with evaluations $c_{e_2}^1, c_{e_3}^2$
For each condition, we dispose of 9 quantiles and 6 evaluations for an expert $e$ at the end of the elicitation process:

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- **Light exercise**: $q_{e2}^{0.25}, q_{e2}^{0.5}, q_{e2}^{0.75}$ with evaluations $c_{e2}^1, c_{e2}^2$
- **Heavy exercise**: $q_{e3}^{0.25}, q_{e3}^{0.5}, q_{e3}^{0.75}$ with evaluations $c_{e2}^1, c_{e3}^2$

⇒ **Idea 1:**
Fit independent beta distributions with parameters $\alpha_{ej}$ and $\beta_{ej}$ via least squared for each variable $j$, denoted $P_{ej}$ hereafter
Results for a hewer after the mechanisation

Individual experts: Beta

Individual experts: Dirichlet

Proportion of time spent sitting

Proportion of time spent in light exercise

Proportion of time spent in heavy exercise
For each condition, we dispose of 9 quantiles and 6 evaluations for an expert e at the end of the elicitation process:

- **Sitting**: \( q_{e_1}^{0.25}, q_{e_1}^{0.5}, q_{e_1}^{0.75} \) with evaluations \( c_{e_1}^1, c_{e_1}^2 \)
- **Light exercise**: \( q_{e_2}^{0.25}, q_{e_2}^{0.5}, q_{e_2}^{0.75} \) with evaluations \( c_{e_2}^1, c_{e_2}^2 \)
- **Heavy exercise**: \( q_{e_3}^{0.25}, q_{e_3}^{0.5}, q_{e_3}^{0.75} \) with evaluations \( c_{e_3}^1, c_{e_3}^2 \)

⇒ **Idea 2:**

Fit marginal beta distributions of a Dirichlet distribution with parameters \( \hat{a}_{e_1}, \hat{a}_{e_2} \) and \( \hat{a}_{e_3} \) so that \( P_{e_1} + P_{e_2} + P_{e_3} = 1 \)

- \( P_{e_1} \sim Beta(\hat{a}_{e_1}, \hat{a}_{e_2} + \hat{a}_{e_3}) \)
- \( P_{e_2} \sim Beta(\hat{a}_{e_2}, \hat{a}_{e_1} + \hat{a}_{e_3}) \)
- \( P_{e_3} \sim Beta(\hat{a}_{e_3}, \hat{a}_{e_1} + \hat{a}_{e_2}) \)
Results for a hewer after the mechanisation

Individual experts: Beta

Individual experts: Dirichlet

Proportion of time spent sitting
Density
0
5
10
15
0.00 0.25 0.50 0.75 1.00
Proportion of time spent in light exercise
Density
0
5
10
15
0.00 0.25 0.50 0.75 1.00
Proportion of time spent in heavy exercise
Density
0
5
10
15
0.00 0.25 0.50 0.75 1.00
Fitting a distribution that reflects an expert’s knowledge

For each condition, we dispose of 9 quantiles and 6 evaluations for an expert e at the end of the elicitation process:

- Sitting: \( q_{e_1}^{0.25}, q_{e_1}^{0.5}, q_{e_1}^{0.75} \) with evaluations \( c_{e_1}^1, c_{e_1}^2 \)
- Light exercise: \( q_{e_2}^{0.25}, q_{e_2}^{0.5}, q_{e_2}^{0.75} \) with evaluations \( c_{e_2}^1, c_{e_2}^2 \)
- Heavy exercise: \( q_{e_3}^{0.25}, q_{e_3}^{0.5}, q_{e_3}^{0.75} \) with evaluations \( c_{e_2}^1, c_{e_3}^2 \)

⇒ Idea 2:
Fit marginal beta distributions of a Dirichlet distribution with parameters \( \hat{a}_{e_1}, \hat{a}_{e_2} \) and \( \hat{a}_{e_3} \) so that \( P_{e_1} + P_{e_2} + P_{e_3} = 1 \)

- \( P_{e_1} \sim \text{Beta}(\hat{a}_{e_1}, \hat{a}_{e_2} + \hat{a}_{e_3}) \)
- \( P_{e_2} \sim \text{Beta}(\hat{a}_{e_2}, \hat{a}_{e_1} + \hat{a}_{e_3}) \)
- \( P_{e_3} \sim \text{Beta}(\hat{a}_{e_3}, \hat{a}_{e_1} + \hat{a}_{e_2}) \)

- Denote \( f_{e} \) the Dirichlet distribution with parameters \( \hat{a}_{e_1}, \hat{a}_{e_2} \) and \( \hat{a}_{e_3} \) derived for expert e with relative weight \( \pi_e = \frac{\sum_{k,l} c_{ek}^l}{\sum_{i,k,l} c_{ik}^l} \)
Combining the information of several experts
Approaches for the aggregation of expert knowledge

- Averaging
- Mixture modeling
- Hierarchical approach
Averaging

Given

- \( f_1 = \text{Dirichlet}(\hat{a}_{11}, \hat{a}_{12}, \hat{a}_{13}), \pi_1 \)
- \( f_2 = \text{Dirichlet}(\hat{a}_{21}, \hat{a}_{22}, \hat{a}_{23}), \pi_2 \)
- \( f_3 = \text{Dirichlet}(\hat{a}_{31}, \hat{a}_{32}, \hat{a}_{33}), \pi_3 \)

derive an averaged distribution \( f = \text{Dirichlet}(a_1, a_2, a_3) \) with

- \( a_1 = \pi_1 \hat{a}_{11} + \pi_2 \hat{a}_{21} + \pi_3 \hat{a}_{31} \)
- \( a_2 = \pi_1 \hat{a}_{12} + \pi_2 \hat{a}_{22} + \pi_3 \hat{a}_{32} \)
- \( a_3 = \pi_1 \hat{a}_{13} + \pi_2 \hat{a}_{23} + \pi_3 \hat{a}_{33} \)
Mixture modeling

Given

- $f_1 = \text{Dirichlet}(\hat{a}_{11}, \hat{a}_{12}, \hat{a}_{13}), \pi_1$
- $f_2 = \text{Dirichlet}(\hat{a}_{21}, \hat{a}_{22}, \hat{a}_{23}), \pi_2$
- $f_3 = \text{Dirichlet}(\hat{a}_{31}, \hat{a}_{32}, \hat{a}_{33}), \pi_3$

derive the mixture model $f = \pi_1 f_1 + \pi_2 f_2 + \pi_3 f_3$
Hierarchical approach

- Derive $f = \text{Dirichlet}(a_1, a_2, a_3)$ with $a_1 = s\theta_1$, $a_2 = s\theta_2$ and $a_3 = s - a_1 - a_2$
- Treat the parameters of the fitted Dirichlet distributions as data
- Reparameterize: $\hat{s}_e = \hat{a}_{e1} + \hat{a}_{e2} + \hat{a}_{e3}$, $\hat{\theta}_{e1} = \frac{\hat{a}_{e1}}{\hat{s}_e}$ and $\hat{\theta}_{e2} = \frac{\hat{a}_{e2}}{\hat{s}_e}$
- Model $\hat{\theta}_{e1}$, $\hat{\theta}_{e2}$ and $\hat{s}_e$ as
  - $\hat{\theta}_{e1}|\theta_1, s, \pi_e \sim \text{Beta}(\theta_1 s\pi_e, (1 - \theta_1)s\pi_e)$
  - $\hat{\theta}_{e2}|\hat{\theta}_{e1}, \theta_2, s, \pi_e \sim \text{Beta}(\theta_2 s\pi_e, (1 - \theta_2)s\pi_e)\mathbb{1}_{\hat{\theta}_{e2} \leq 1 - \hat{\theta}_{e1}}$
  - $\hat{s}_e | s, \pi_e \sim \text{Gamma}(\pi_e, \pi_e/s)$
- Assume the following prior distributions:
  - $\theta_1 \sim \text{Unif}(0, 1)$
  - $\theta_2|\theta_1 \sim \text{Unif}(0, 1)\mathbb{1}_{\theta_2 \leq 1 - \theta_1}$
  - $s \sim \text{Exp}(0.01)$
Hierarchical approach

- Derive \( f = \text{Dirichlet}(a_1, a_2, a_3) \) with \( a_1 = s\theta_1, a_2 = s\theta_2 \) and \( a_3 = s - a_1 - a_2 \)
- Model \( \hat{\theta}_{e1}, \hat{\theta}_{e2} \) and \( \hat{s}_e \) as
  - \( \hat{\theta}_{e1}|\theta_1, s, \pi_e \sim \text{Beta}(\theta_1 s\pi_e, (1 - \theta_1)s\pi_e) \)
  - \( \hat{\theta}_{e2}|\hat{\theta}_{e1}, \theta_2, s, \pi_e \sim \text{Beta}(\theta_2 s\pi_e, (1 - \theta_2)s\pi_e)\mathbb{1}_{\hat{\theta}_{e2} \leq 1 - \hat{\theta}_{e1}} \)
  - \( \hat{s}_e|s, \pi_e \sim \text{Gamma}(\pi_e, \pi_e/s) \)
- Assume the following prior distributions:
  - \( \theta_1 \sim \text{Unif}(0, 1) \)
  - \( \theta_2|\theta_1 \sim \text{Unif}(0, 1)\mathbb{1}_{\theta_2 \leq 1 - \theta_1} \)
  - \( s \sim \text{Exp}(0.01) \)

\[ E(\hat{\theta}_{e1}) = \theta_1, E(\hat{\theta}_{e2}) = \theta_2, E(\hat{s}_e) = s \]
\[ \text{Var}(\hat{\theta}_{e1}) = \frac{\theta_1(1 - \theta_1)}{s\pi_e + 1}, \text{Var}(\hat{\theta}_{e2}) = \frac{\theta_2(1 - \theta_2)}{s\pi_e + 1}, \text{Var}(\hat{s}_e) = \frac{s^2}{\pi_e} \]
Results for a hewer after the mechanisation
Derive a prior distribution on breathing rate

\[
\tilde{br}_i(t) = P_{1i}(t) \cdot br_1 + P_{2i}(t) \cdot br_2 + P_{3i}(t) \cdot br_3
\]

- **ICRP**
  - Publication 66:
    - 1.0 - 1.7 m³h⁻¹
- **Alpha risk**
  - Project:
    - 0.9 - 1.7 m³h⁻¹
    - 1.1 - 2.1 m³h⁻¹
The hierarchical approach for the aggregation of the opinion of several experts allows to combine the information derived by expert opinion with the information available in the literature.

Integrate the prior elicitation sub-model in the hierarchical model to account for exposure and dose uncertainty.
Perspectives

- \( \lambda \)
- \( \beta \)
- \( \theta_1 \)
- \( \theta_2 \)
- \( s \)
- \( \mu_X \)
- \( \sigma_X \)
- \( \sigma_{\epsilon^B} \)
- \( \sigma_{\epsilon^C} \)

- \( V_i(t) \): additional covariates
- \( D_i^{\text{cum}}(t) \): cumulated dose
- \( Y_i, \delta_i \): time until death by lung cancer
- \( \hat{\theta}_{1e} \)
- \( \hat{\theta}_{2e} \)
- \( \hat{\sigma}_e \)
- \( br_i(t) \)
- \( X_i(t) \): true annual exposure
- \( Z_i^B(t) \): observed annual exposure (Berkson period)
- \( Z_i^C(t) \): observed annual exposure (classical period)

- True annual cumulated dose
- Additional covariates

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13/06/2017 27 / 28
Thank you for your attention